Exercise 36

Prove that $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$.

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad 0 < |x - 2| < \delta \qquad \text{then} \qquad \left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x - 2|$.

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$$

$$\left|\frac{2 - x}{2x}\right| < \varepsilon$$

$$\left|\frac{1}{2x}(2 - x)\right| < \varepsilon$$

$$\left|\frac{1}{2x}\right| |x - 2| < \varepsilon$$

$$\frac{1}{|2x|}|x - 2| < \varepsilon$$

On an interval centered at $x = 2$, a positive constant C can be chosen so that $1/|2x| < C$.

$$C|x - 2| < \varepsilon$$

$$|x - 2| < \frac{\varepsilon}{C}$$

To determine C, suppose that x is within a distance a from 2.

$$|x - 2| < a$$

$$-a < x - 2 < a$$

$$2 - a < x < 2 + a$$

$$4 - 2a < 2x < 4 + 2a$$

$$\frac{1}{|4 - 2a|} > \frac{1}{|2x|}$$

The constant C is then $1/|4 - 2a|$, which means $\varepsilon/C = |4 - 2a|\varepsilon$.

Choose δ to be whichever is smaller between a and $|4-2a|\varepsilon$: $\delta = \min\{a, |4-2a|\varepsilon\}$. Now, assuming that $|x-2| < \delta$,

$$\begin{aligned}
\left|\frac{1}{x} - \frac{1}{2}\right| &= \left|\frac{2-x}{2x}\right| \\
&= \left|\frac{1}{2x}(2-x)\right| \\
&= \left|\frac{1}{2x}\right| |x-2| \\
&= \frac{1}{|2x|}|x-2| \\
&< C\delta \\
&= \left(\frac{1}{|4-2a|}\right)(|4-2a|\varepsilon) \\
&= \varepsilon.
\end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}.$$